$$\Delta t = \left[\frac{2+k^2}{c(\mu k-2)}\right]^{1/2} \ln\left(1+\frac{4\,|\xi_0^{\cdot}|\sqrt{c(\mu k-2)}}{\mu\,|r_1(P_2-P_1)|\,k\sqrt{2+k^2}}\right)$$

Note that the same result could have been obtained by applying the formulae and theorems derived from the foregoing general analysis.

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PMM U.S.S.R., Vol.54, No.4, pp. 438-444, 1990
Printed in Great Britain

0021-8928/90 $10.00+0.00

SEPARATION OF MOTIONS IN NON-LINEAR OSCILLATORY SYSTEMS WITH RANDOM PERTURBATIONS*

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An asymptotic procedure is developed for the separation of motions in non-linear stochastic systems which are reducible to standard form with rotating phase. It is shown that the slowly varying component of the motion can be approximated by a diffusion process. An example of a body moving in a periodic force field under the action of random disturbances is studied.

Previous publications /1-3/ have investigated the dynamics of randomly perturbed systems which are reducible to standard form

$$x^{\cdot} = \varepsilon F(x, \xi(t)) + \varepsilon^2 G(x, \xi(t)),\; x(0) = a \in R_n \tag{0.1}$$

Here $\xi(t)$ is a stochastic process with values in R_l, and ε is a small parameter. It was proved that if the coefficients of the system satisfy certain conditions (the most general statement of which may be found in /3/), the solution $x(t, \varepsilon)$ of system (0.1) is weakly convergent /4/ to a diffusion process $x_0(\tau)$ - the solution of the stochastic differential equation

$$dx_0 = b(x_0)d\tau + \sigma(x_0)dw,\; x_0(0) = a;\; \tau = \varepsilon^2 t \tag{0.2}$$

where $w(\tau)$ is an l-dimensional standard Wiener process, and the coefficients b and σ are evaluated by averaging certain moment characteristics of the coefficients of system (0.1). In other words, one can identify a "slow" diffusion component in the motion of system (0.1), upon which small (in the weak sense) and rapid perturbations are superimposed. Considerable efforts have been made in the literature to justify the passage to the limit from (0.1) to (0.2); a detailed bibliography may be found in /3/. Applications of this approach to some problems of stochastic dynamics in non-linear oscillatory systems are discussed in /5, 6/.

*Prikl. Matam. Mekhan., 54, 4, 530-536, 1990

1. We shall obtain results analogous to those presented in /3/ for systems whose dynamics are described by the equations

$$x^{\cdot} = \varepsilon F(x, \theta, \xi(\theta)) + \varepsilon^2 G(x, \theta),\ x(0) = a \in R_n \tag{1.1}$$
$$\theta^{\cdot} = \omega(x) + \varepsilon H(x, \theta, \xi(\theta)) + \varepsilon^2 D(x, \theta),\ \theta(0) = 0 \in R_1$$

To construct the solution we shall employ the asymptotic procedure of diffusion approximations /7/, developed in /3/ for analysing systems of type (0.1).

We recall the necessary definitions from /7/. Let $f(t)$ be a stochastic process with values in R_1, defined in a probability space /8/ (for brevity we shall indicate the dependence of f on time only). Now let $M_s f(t)$ be the conditional expectation of the process $f(t)$ given $s \leqslant t$ /8/. It is assumed that $f(t)$ is right continuous, vanishes outside some finite interval $t \in [0, T]$ and $\sup_t M|f(t)| < \infty$. If $f(t)$ has these properties, we write $f(t) \in \Phi$.

We now introduce the operator L^ε and its domain of definition $D(L^\varepsilon)$ /7/. We shall say that $f \in D(L^\varepsilon)$ and $L^\varepsilon f = g$, if $f, g \in \Phi$ and

$$\lim_{\delta \to 0+} M|\delta^{-1}[M_\tau f(\tau + \delta) - f(\tau)] - g(\tau)| = 0 \tag{1.2}$$

From (1.2) it follows that /3, 7/

$$M_\tau f(\tau + \delta) - f(\tau) = \int_\tau^{\tau+\delta} M_\tau L^\varepsilon f(u)\, du \tag{1.3}$$

In particular, if $f(\tau) = f(x_\varepsilon(\tau))$, where $x_\varepsilon(\tau)$ is a solution of some perturbed system, then formula (1.3) indicates a way of calculating the functional $M_\tau f(x_\varepsilon(\tau + \delta))$ on paths of the system. If $f(\tau) = f(x_0(\tau))$, where $x_0(\tau)$ is a solution of Eq.(0.2), then /3, 7, 8/

$$L^\varepsilon = L = b(x)\frac{\partial}{\partial x} + \frac{1}{2}\operatorname{Tr} A(x)\frac{\partial^2}{\partial x^2},\quad A = \sigma\sigma' \tag{1.4}$$

$$M_\tau f(x_0(\tau + \delta)) - f(x_0(\tau)) = \int_\tau^{\tau+\delta} M_\tau L f(x_0(u))\, dw \tag{1.5}$$

Relations (1.3) and (1.5) indicate a way of calculating and comparing the functionals on the paths of the perturbed and diffusion systems. As shown in /7/, if, given any sufficiently smooth function $f(x)$ of compact support, one can find a function $f^\varepsilon(\tau)$ such that

$$\lim_{\varepsilon \to 0} M|f^\varepsilon(\tau) - f(x_\varepsilon(\tau))| = 0 \tag{1.6}$$
$$\lim_{\varepsilon \to 0} M|L^\varepsilon f^\varepsilon(\tau) - Lf(x_\varepsilon(\tau))| = 0,\quad \tau \in [0, T]$$

and for $\varepsilon \in (0, \varepsilon_0]$, $\tau \in [0, T]$ the sequence $x_\varepsilon(\tau)$ is weakly compact /4/, then the process $x_\varepsilon(\tau)$ converges weakly as $\varepsilon \to 0$ to the diffusion process $x_0(\tau)$ with $x_0(0) = x_\varepsilon(0) = a$.

Different constructions of approximating operators L have been proposed /1-3/ for systems of type (0.1). Using the technique of /3/, we shall construct a suitable operator for a system of type (1.1).

2. Henceforth we assume that $\omega(x) \geqslant \omega_0 > 0$, $MF(x, \theta, \xi(\theta)) = MH(x, \theta, \xi(\theta)) = 0$ for fixed x. Other restrictions on the coefficients of system (1.1) will be specified as the need arises.

Put $\tau = \varepsilon^2 t$, and let $x(t, \varepsilon) = x_\varepsilon(\tau)$, $\theta(t, \varepsilon) = \theta_\varepsilon(\tau)$ be a solution of system (1.1). Define $f(\tau, x)$, to be a sufficiently smooth function on paths of system (1.1), vanishing outside some bounded region $D: \{x \in S, \tau \in [0, T]\}$. Let $f^\varepsilon(\tau)$ be a function related to $f(\tau, x_\varepsilon(\tau))$ by

$$f^\varepsilon(\tau) = f(\tau, x_\varepsilon) + \varepsilon f_1(\tau, x_\varepsilon, \theta_\varepsilon) + \varepsilon^2 f_2(\tau, x_\varepsilon, \theta_\varepsilon) \tag{2.1}$$

where $x_\varepsilon = x_\varepsilon(\tau), \theta_\varepsilon = \theta_\varepsilon(\tau)$, with the coefficients f_1, f_2 so chosen as to satisfy (1.6).

Following /3/, we write

$$\begin{aligned} L^\varepsilon f^\varepsilon(\tau) = {} & \varepsilon^{-1}(f_x'F + \omega L_\theta^\varepsilon f_1) + (f_\tau + f_{1x}'F + f_x'G + HL_\theta^\varepsilon f_1 + \omega L_\theta^\varepsilon f_2) + \\ & \varepsilon(f_{1\tau} + f_{2x}'F + f_{1x}'G + DL_\theta^\varepsilon f_1 + HL_\theta^\varepsilon f_2) + \\ & \varepsilon^2(f_{2x}'G + f_{2\tau} + DL_\theta^\varepsilon f_2) + \varepsilon^3 \ldots \end{aligned} \tag{2.2}$$

the prime denotes transposition; function arguments are omitted. The operator L_θ^ε is defined by analogy with L^ε:

$$L_\theta^\varepsilon f = \lim_{\Delta\to 0} \Delta^{-1}[M_\theta f(\tau, x, \theta+\Delta) - f(\tau, x, \theta)] \tag{2.3}$$

where the arguments τ, x are considered here as fixed parameters. The equality is understood in the weak sense of (1.2).

Let us construct the function f_1 in such a way as to make the coefficient of ε^{-1} vanish (all equalities are understood in the weak sense). Choose f_1 in the form

$$f_1(\tau, x, \theta) = \omega^{-1}(x) f_x'(\tau, x) \int_\theta^\infty \mathrm{M}_\theta F(x, u, \xi(u))\, du \tag{2.4}$$

Then, by definition (2.3),

$$L_\theta^\varepsilon f_1(\tau, x, \theta) = \omega^{-1}(x) f_x'(\tau, x) \lim_{\delta\to 0+} \Big\{\delta^{-1}\Big[\mathrm{M}_\theta \int_{\theta+\delta}^\infty \mathrm{M}_{\theta+\delta} F(x, u, \xi(u))\, du - \int_\theta^\infty \mathrm{M}_\theta F(x, u, \xi(u))\, du\Big]\Big\}$$

It follows from the properties of conditional expectations /8/ that

$$L_\theta^\varepsilon f_1(\tau, x, \theta) = -\omega^{-1}(x) f_x'(\tau, x) \lim_{\delta\to 0+} \int_\theta^{\theta+\delta} \mathrm{M}_\theta F(x, u, \xi(u))\, du = -\omega^{-1}(x) f_x'(\tau, x) F(x, \theta, \xi(\theta))$$

i.e., the first term in (2.2) vanishes. The function f_2 is constructed in such a way that the second term in (2.2) does not contain secular terms in θ. We define

$$f_2 = \omega^{-1}(x) \sum_{j=1}^2 [I_j(\tau, x, \theta) - S_j(\tau, x, \theta)] \tag{2.5}$$

$$I_j = \int_\theta^\infty [\mathrm{M}_\theta Q_j(\tau, x, u) - \mathrm{M} Q_j(\tau, x, u)]\, du, \quad S_j = \int_0^\theta [\mathrm{M} Q_j(\tau, x, u) - \bar{Q}_j(\tau, x)]\, du$$

$$Q_1 = f_\tau(\tau, x) + f_{1x}'(\tau, x, u) F(x, u, \xi(u)) = f_\tau(\tau, x) + \int_u^\infty \mathrm{M}_u [f_x'(\tau, x) F_1(x, z, \xi(z))]_x'\, dz F(x, u, \xi(u)) \tag{2.6}$$

$$Q_2 = f_x'(\tau, x)[-F_1(x, u, \xi(u)) H(x, u, \xi(u)) + G(x, u)]$$

$$\bar{Q}_j(\tau, x) = \lim_{T\to\infty} \frac{1}{T} \int_0^T \mathrm{M} Q_j(\tau, x, u)\, du, \quad F_1 = \omega^{-1} F \tag{2.7}$$

It is assumed that the limits (2.7) exist uniformly in τ, $x \in D$.

Substituting (2.4) and (2.6) into (2.2), we obtain

$$L^\varepsilon f^\varepsilon(\tau) = f_\tau(\tau, x_\varepsilon) + \sum_{j=1}^2 [\bar{Q}_j(\tau, x_\varepsilon) + \varepsilon^j R_j(\tau, x_\varepsilon, \theta_\varepsilon)] \tag{2.8}$$

Here R_1, R_2 are the coefficients of $\varepsilon, \varepsilon^2$ in (2.2). All terms on the right of (2.8) may be treated as operators acting on $f(\tau, x_\varepsilon(\tau))$, $x_\varepsilon = x_\varepsilon(\tau)$, $\theta_\varepsilon = \theta_\varepsilon(\tau)$.

Let us change the form of the principal terms of the expansion (2.8). Utilizing the properties of the conditional expectation /8/, we obtain

$$\bar{Q}_1(\tau, x) = \lim_{T\to\infty} \frac{1}{T} \int_0^T du \int_u^\infty \mathrm{M}\{[f_x'(\tau, x) F_1(x, z, \xi(z))]_x' F(x, u, \xi(u))\}\, dz$$

For sufficiently smooth F, ω, we have

$$\bar{Q}_1(\tau, x) = K_1'(x) f_x(\tau, x) + \frac{1}{2} \operatorname{Tr} A(x) f_{xx}(\tau, x) \tag{2.9}$$

$$K_1(x) = \lim_{T\to\infty} \frac{1}{T} \int_0^T d\theta \int_\theta^\infty K(x, \theta, u)\, du \tag{2.10}$$

$$A(x) = \lim_{T\to\infty} \frac{1}{T} \int_0^T d\theta \int_0^\infty A(x, \theta, u)\, du$$

$$K(x, \theta, u) = \mathrm{M}[F_{1x}(x, u, \xi(u))\, F(x, \theta, \xi(\theta))]$$

$$A(x, \theta, u) = M[F_1(x, u, \xi(u))\, F'(x, \theta, \xi(\theta))]$$

Accordingly,

$$\bar{Q}_2(\tau, x) = K_2'(x) f_x(\tau, x),\ K_2(x) = \lim_{T\to\infty} \frac{1}{T} \int_0^T MK_2(u, x)\, du \tag{2.11}$$

where $K_2(u, x)$ is the coefficient of f_x in formula (2.6). Thus,

$$L^\varepsilon f^\varepsilon(\tau) = f_\tau(\tau, x_\varepsilon) + Lf(\tau, x_\varepsilon) + \sum_{j=1}^{2} \varepsilon^j R_j(\tau, x_\varepsilon, \theta_\varepsilon) \tag{2.12}$$

$$L = b'(x) \frac{\partial}{\partial x} + \frac{1}{2} \operatorname{Tr} A(x) \frac{\partial^2}{\partial x^2}, \quad b = K_1 + K_2 \tag{2.13}$$

Following /3/, it can be shown that conditions (1.6) will hold if $\mathrm{M}\,|\,f_j(\tau, x, \theta)\,| < \infty$, $\mathrm{M}\,|\,R_j(\tau, x, \theta)\,| < \infty$ for $\theta \in (-\infty, \infty), \tau, x \in D, j = 1, 2$. To estimate the relevant terms, we will specify in more detail the restrictions on the coefficients of the system. We shall assume throughout the sequel that the following conditions are satisfied: F, H may be expressed as

$$F(x, \theta, \xi(\theta)) = F_0(x, \theta)\, \xi(\theta),\ H(x, \theta, \xi(\theta)) = H_0(x, \theta)\, \xi(\theta) \tag{2.14}$$

where the random peturbations $\xi(\theta)$ belong to one of two types (Conditions A):

1) $\xi(\theta)$ is a stationary right continuous normal Markov process with zero mean;

2) $\xi(\theta)$ is an almost surely bounded stationary process with zero mean, satisfying a uniformly strong mixing condition /4/ with coefficient $\varphi(u)$ such that

$$\int_0^\infty \varphi^{1/2}(u)\, du < \infty$$

The coefficients $U = (F_0, H_0, G, D, \omega)$ satisfy the following conditions (Conditions B):

1) the functions U are bounded and periodic or uniformly quasiperiodic in θ for $\theta \in (-\infty, \infty)$ uniformly in $x \in S$;

2) the functions U are continuous in x for all $x \in R_n$ uniformly in $\theta \in (-\infty, \infty)$;

3) the derivatives of the functions U and the second derivatives of the functions F_0, ω with respect to x are continuous for $x \in R_n$ and bounded for $x \in S$ uniformly in $\theta \in (-\infty, \infty)$;

4) the limits (2.10), (2.11) exist uniformly in x for $x \in S$.

Let us clarify Conditions A for the case in which $\xi(\theta)$ is a stationary normal process. Calculating the conditional expectation, we obtain /8/

$$\mathrm{M}_\theta \xi(u) = \chi(\theta - u)\, \xi(\theta), \quad \chi(\theta) = K_\xi(\theta)/K_\xi(0) \tag{2.15}$$

where $K_\xi(\theta)$ is the correlation function of the process $\xi(\theta)$, and

$$\int_0^\infty |\chi(\theta)|^p\, d\theta < \infty, \quad p > 0 \tag{2.16}$$

Substituting (2.14)-(2.16) into (2.4), we obtain $\mathrm{M}\,|\,f_1(\tau, x, \theta)\,| < \infty$. Similar arguments lead to the conclusion that if Conditions A and B are satisfied, then

$$\mathrm{M}\,|\,R_j(\tau, x, \theta)\,| < \infty, \quad \mathrm{M}\,|\,I_j(\tau, x, \theta)\,| < \infty, j = 1, 2 \tag{2.17}$$

in the region indicated above. Detailed estimates of the conditional expectations may be found in /3/. An analogous estimate for the deterministic terms

$$|\,S_j(\tau, x, \theta)\,| < \infty, \quad j = 1, 2 \tag{2.18}$$

follows directly from condition 4 and may be constructed in the same way as for deterministic systems /9/.

Thus, conditions (1.6) are satisfied. Weak compactness of the sequence $x_\varepsilon(\tau)$ is proved using the same arguments as for systems in standard form /3/.

Let $x_0(\tau)$ be a solution of Eq.(0.2) for the operator L defined by (2.9)-(2.13). It follows from Conditions B that the coefficients of L are continuous and continously differentiable with respect to x in any compact set $K \subset R_n$. Let us assume additionally that L is uniformally parabolic /10/ in the region of interest and that the process $x_0(\tau)$ satisfies the regularity conditions /11/. Then /11/ a unique solution $x_0(\tau)$ of the limiting Eq.(0.2) exists. It was proved in /3, 7/ that (1.6) implies that the finite-dimensional distributions of the processes $x_\varepsilon(\tau)$ and $x_0(\tau)$ are convergent and, as a corollary, that $x_\varepsilon(\tau)$ and $x_0(\tau)$ are weakly convergent.

For functionals of the type $\Phi_\varepsilon = M_{0,a}\varphi(x_\varepsilon(\tau_f))$ it is convenient to construct a direct estimate of the proximity of Φ_ε to $\Phi_0 = M_{0,a}\varphi(x_0(\tau_f))$, $\tau_f \in [0, T]$.

Let $f(\tau, x)$ be a solution of the Cauchy problem

$$\partial f/\partial\tau + Lf = 0, \quad f(\tau_f, x) = \varphi(x) \tag{2.19}$$

Let $\varphi(x) \in C_4$ be a function with compact support defined for $x \in K$. If the operator L has the properties described above,then a solution of problem(2.19)exists and $f(\tau, x) \in C_{2,4}$ for $\tau \in [0, T]$, $x \in K$ /10/. Since $x_0(\tau)$ is a regular process, one can always choose T and K in such a way that $x_0(\tau) \in K$ for all $0 \leqslant \tau \leqslant \tau_f \leqslant T$ provided that $x_0(0) = a \in \operatorname{int} K$ /11/. Under these conditions, $f(\tau, x) = M_{\tau,x}\varphi(x_0(\tau_f))$ /8, 11/.

If the process $x_\varepsilon(\tau)$ is continuous, a number τ_K exists such that $x_\varepsilon(\tau) \in K$ for $0 \leqslant \tau \leqslant \tau_K$, $x_\varepsilon(0) = a$. Consequently, when $0 \leqslant \tau \leqslant \tau_f \leqslant \tau_K$ all the constructions of Sect.2 remain valid.

Let $x_\varepsilon(\tau) = x$, $\theta_\varepsilon(\tau) = \theta$, $(x, \theta) = y \in R_{n+1}$. Then, in view of (1.5), (2.1) and (2.12), we can write

$$\begin{gathered} M_{\tau,x} f(\tau_f, x_\varepsilon(\tau_f)) - f(\tau, x) + \varepsilon M_{\tau,y} F(\tau_f, y_\varepsilon(\tau_f), \varepsilon) = \\ \int_\tau^{\tau_f} M_{\tau,x}[f_u(u, x_\varepsilon(u)) + Lf(u, x_\varepsilon(u))]\,du + \varepsilon \int_\tau^{\tau_f} M_{\tau,y} R(u, y_\varepsilon(u), \varepsilon)\,du \\ y_\varepsilon(u) = (x_\varepsilon(u), \theta_\varepsilon(u)) \in R_{n+1}, \\ F(\tau, y, \varepsilon) = f_1(\tau, x, \theta) + \varepsilon f_2(\tau, x, \theta), \quad R(\tau, y, \varepsilon) = R_1(\tau, x, \theta) + \\ \varepsilon R_2(\tau, x, \theta) \end{gathered} \tag{2.20}$$

By estimates (2.17) and (2.18).

$$|M_{\tau,x} f(\tau_f, x_\varepsilon(\tau_f)) - f(\tau, x)| \leqslant \varepsilon[C_1 + C_2(\tau_f - \tau)] \tag{2.21}$$

where $C_1 > 0$, $C_2 > 0$ are constants independent of ε. At the same time, it follows from (2.19) that $f(\tau_f, x_\varepsilon(\tau_f)) = \varphi(x_\varepsilon(\tau_f))$. Thus,

$$|M_{\tau,x}\varphi(x_\varepsilon(\tau_f)) - M_{\tau,x}\varphi(x_0(\tau_f))| \leqslant \varepsilon[C_1 + C_2(\tau_f - \tau)] \tag{2.22}$$

Putting $\tau = 0$, $x = a$, we get

$$|\Phi_\varepsilon - \Phi_0| \leqslant \varepsilon(C_1 + C_2\tau) \tag{2.23}$$

Using the regularity of $x_0(\tau)$ and Chebyshev's inequality, it can be shown that estimates (2.22), (2.23) imply regularity for the process $x_\varepsilon(\tau)$ (the proof is carried out along the same lines as in /11, Chap.3, Sect.4/). Estimate (2.23) remains valid for all finite values of τ_f.

Remarks. 1. For systems in standard form, one must put $\theta = t$, $\theta^\cdot = 1$. When that is done formulae (2.9)-(2.11) reduce to standard formulae /1/.

2. All the transformations remain true if the coefficients of the system have the form $F = F_1(x, \theta, \xi(\theta)) + F_2(x, t, \zeta(t))$ and so on, where the random perturbations $\xi(\theta)$ and $\zeta(t)$ are indepenent. In the expansion (2.1) we must then put $f_j = f_j^1(\tau, x, \theta) + f_j^2(\tau, x, t)$, $j = 1, 2$ and construct the functions f_j^1 and f_j^2 by the above rules, independently of one another.

3. If the coefficients depend on slow time $\tau = \varepsilon^2 t$, then all transformations remain true. In that case τ may be considered as an additional slow variable defined by the equation $\tau^\cdot = \varepsilon^2$. For systems in standard form the results are identical with those obtained in /2, 3/.

3. Consider the following model example: the motion of a point in a weak force field. Suppose that the equations of motion, allowing for the comparative smallness of the disturbing the driving factors, reduce to the form

$$\begin{gathered} \theta^{\cdot\cdot} = \varepsilon^2 f(\theta) - \varepsilon[\varepsilon b + \xi(\theta)]\theta^\cdot + \varepsilon[\varepsilon u(t) + \eta(t)]; \quad \theta(0) = 0, \\ \theta^\cdot(0) = \gamma; \quad b > 0 \end{gathered} \tag{3.1}$$

or to the analogous form with $\xi(\theta)$ replaced by $\zeta(t)$.

Here θ is the coordinate of the point, $f(\theta)$ is a 2π-periodic function characterizing the force field, $u(t)$ is a T-periodic function characterizing the external energy source, and ξ, ζ and η are perturbing factors. The perturbation $\xi(\theta)$ usually denotes the resistance of the external medium, $\zeta(t)$ denotes fluctuations in the damping coefficient in the damping mechanisms, and $\eta(t)$ denotes fluctuations in the external load. The different powers of the small parameters mean that the effects of the deterministic and random factors are taken into account to within the same accuracy (see (2.6)).

Let us see how the nature of the scattering of dissipative forces affects the dynamics of the system. Reducing (3.1) to standard form, we have

$$\theta^{\cdot} = x, \quad \theta(0) = 0$$
$$x^{\cdot} = \varepsilon[-\xi(\theta)x + \eta(t)] + \varepsilon^2[u(t) + f(\theta) - bx], \quad x(0) = \gamma \tag{3.2}$$

Let us assume from now on that

$$\frac{1}{2\pi}\int_0^{2\pi} f(\theta)\,d\theta = \bar{f}, \quad \frac{1}{T}\int_0^T u(t)\,dt = \bar{u}, \quad \bar{f} + \bar{u} = u_0$$

We shall also assume that $\xi(\theta)$, $\zeta(t)$ and $\eta(t)$ are independent stochastic processes satisfying Conditions A. Since the perturbations are independent, all the arguments of Sect. 2 remain valid, provided that the averaging is performed with respect to the appropriate argument. For Eq.(3.2) we have

$$K(x) = 0, \quad A(x) = a^2(x) \tag{3.3}$$
$$a^2(x) = \frac{1}{2\pi}\int_0^{2\pi} d\theta \int_{-\infty}^{\infty} \alpha_\xi(x, \theta, u)\,du + \frac{1}{T}\int_0^T dt \int_{-\infty}^{\infty} \alpha_\eta(x, t, s)\,ds$$
$$\alpha_\xi = \mathrm{M}[\xi(\theta)\xi(u)]x = K_\xi(\theta - u)x, \quad \alpha_\eta = \mathrm{M}[\eta(t)\eta(s)] = K_\eta(t - s)$$
$$a^2(x) = a_\eta^2 + a_\xi^2 x, \quad a_\eta^2 = S_\eta(0), \quad a_\xi^2 = S_\xi(0)$$

Here K_ξ, K_η, K_ζ are the correlation functions and S_ξ, S_η, S_ζ the spectral densities of the processes $\xi(\theta)$, $\eta(t)$ and $\zeta(t)$, respectively.

Thus, the solution $x(t, \varepsilon)$ of Eq.(3.2) is weakly convergent to a solution $x_0(\tau)$ of the stochastic differential equation

$$dx_0 = (u_0 - bx_0)\,d\tau + a(x_0)\,dw, \quad \tau = \varepsilon^2 t \tag{3.4}$$

It follows from (3.4) that the average velocity of the point $\mathrm{M}x \to \mathrm{M}x_0(\tau) = \omega_0$, where ω_0 is a solution of the unperturbed system

$$d\omega_0/d\tau = u_0 - b\omega_0, \quad \omega_0(0) = \gamma \tag{3.5}$$

As it turns out, a more significant characteristic is the mean-square value of the velocity $\mathrm{M}x^2 \to \mathrm{M}x_0^2(\tau)$. The value of the functional $\Phi_0 = \mathrm{M}_{0,\gamma}x_0^2(\tau)$ is determined by solving problem (2.19); the coefficients of L are determined by (3.3) and (3.4). After obvious reduction we obtain

$$f(\sigma, x) = P_0(\sigma) + P_1(\sigma)x + P_2(\sigma)x^2 \tag{3.6}$$
$$P_0(\sigma) = -u_0(2u_0 + a_\xi^2)b^{-2}[(e^{bs} - 1) - {}^1/_2(e^{2bs} - 1)] - {}^1/_2 a_\eta^2 b^{-1}(e^{2bs} - 1),$$
$$P_1(\sigma) = (2u_0 + a_\xi^2)b^{-1}(e^{bs} - e^{2bs}), \quad P_2(\sigma) = e^{2bs}, \quad s = \sigma - \tau$$

Putting $\sigma = 0$ and $x = \gamma$, we obtain the mean-square value of the velocity at time The steady-state mean-square velocity (i.e., at $\tau \to \infty$) is

$$\bar{\Phi}_0 = {}^1/_2 b^{-2}[u_0(2u_0 + a_\xi^2) + ba_\eta^2] \tag{3.7}$$

Strictly speaking, the convergence as $\tau \to \infty$ must be established on a rigorous basis, as in systems in standard form /2, 3/; nevertheless, Eq.(3.7) can be used to estimate the comparative effect of the disturbing factors on the behaviour of the system. Thus, it follows from (3.6) and (3.7) that the perturbation $\xi(\theta)$ does not affect the stability of the system. Moreover, at $u_0 = 0$ the perturbation $\xi(\theta)$ has no effect whatever on the behaviour of the system, provided that τ is sufficiently large.

Consider system (3.1) with $\xi(\theta)$ replaced by $\zeta(t)$. Repeating all the previous arguments, we see that the process $\theta^{\cdot} = x$ is weakly convergent to a solution of the equation

$$dx_0 = (u_0 - \beta x_0)\,d\tau + a_1(x_0)\,dw, \quad x_0(0) = \gamma \tag{3.8}$$
$$(\beta = b - a_\zeta^2/2, \quad a_1^2 = a_\eta^2 + a_\xi^2 x^2, \quad a_\zeta^2 = S_\zeta(0))$$

Thus, the mean velocity is such that $\mathrm{M}x \to \omega_0$, where ω_0 is a solution of the equation obtained from (3.5) by replacing b by β.

The mean-square value of the velocity $\Phi_0 = M_{0,\gamma}x_0^2(\tau)$ has the form (3.6). The coefficient P_0, P_1 P_2, are determined by the formulae

$$P_0(\sigma) = \frac{2u_0}{2\beta_1 - \beta}\left[\frac{1}{\beta}(e^{\beta s} - 1) - \frac{1}{2\beta_1}(e^{2\beta_1 s} - 1)\right] - \frac{a_\eta^2}{2\beta_1}(e^{2\beta_1 s} - 1),$$
$$P_1(\sigma) = \frac{2u_0}{2\beta_1 - \beta}(e^{\beta s} - e^{2\beta_1 s}), \quad P_2(\sigma) = e^{2\beta_1 s}; \quad \beta_1 = b - a_\zeta^2, \quad s = \sigma - \tau$$

Letting $\tau \to \infty$, we get

$$\bar{\Phi}_0 = {}^1/_2 (\beta\beta_1)^{-1} (2u_0^2 + \beta a_\eta^2), \quad \beta > 0, \ \beta_1 > 0$$

Thus, the system is unstable if $b < a_\zeta^2/2$. The variance of the velocity at $\tau \to \infty$, $u_0 = 0$, also depends on a_ζ. Consequently, the nature of the scattering of dissipative forces has a considerable effect on the dynamics of the system.

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Translated by D.L.

PMM U.S.S.R., Vol.54, No.4, pp.444-448, 1990
Printed in Great Britain

0021-8928/90 $10.00+0.00

THE SELFCONSISTENT PROBLEM OF THE VIBRATIONS OF AN INFINITE STRING LOADED WITH A MOVING POINT MASS*

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The problem of the vibrations of a homogeneous infinite string loaded with a point mass, moving in accordance with an unknown law of motion, is considered. This is one of the simplest model selfconsistent problems (SPs) in the dynamics of one-dimensional distributed loaded Lagrangian systems /1/. A mathematical formulation of the problem is given and the conditions for the existence and uniqueness of a global solution are established. An analytical method, which in many cases produces an exact solution, is presented. As an illustration, the displacement of a point mass along a vibrating string, set in motion by an impulse communicated to the mass, is considered. Certain effects related to the reverse action of the radiation of the moving point mass (braking by the radiation) are explained.

**Prikl. Matem. Mekhan.*, 54, 4, 537-542, 1990